

PRICE DEPENDENCE IN OPTIMAL INVESTMENT

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Abstract

We consider a semi-static market composed of derivative securities, which we assume can be traded only at time zero, and of stocks, which can be traded continuously in time. Using a general utility function defined on the positive real line, we study the dependence on the price of the derivatives of the outputs of the utility maximization problem, investigating not only stability but also differentiability, monotonicity, convexity and asymptotic properties.

A classical problem in financial economics is to understand the behavior of rational agents faced with an uncertain evolution of asset prices. In one of the most popular frameworks, one considers an investor who wants to maximize his expected utility from terminal wealth by investing in a frictionless market. This approach takes as inputs a utility function and a model for the future evolution of the stock prices; to implement this program in practice, typically one chooses a particular parametrized family of utility functions and of models for the stock price, and calibrates the value of the parameters to the available data.

Since the choice of the agent's utility and of the market model requires estimation, it is natural to ask how the agent's behavior is affected by misspecifications of the utility function and of the underlying market model. Indeed, following Hadamard's prescription, after investigating existence and uniqueness one should perform stability analysis, and only a problem whose solution exists, is unique and depends continuously on the initial data is worthy of the appellation 'well posed'. This is particularly relevant for numerical implementations of continuous-time models of security trading, where it is important to have convergence of discrete-time to continuous-time models: see e.g. He [He91], Duffie and Protter [DP92] and the monograph Prigent [Pri03].

In fact, multiple papers have recently been published on the topic of sensitivity of the solution of the problem of expected utility maximization under perturbations of various initial conditions. Jouini and Napp [JN04], Carasus and Rásonyi [CR07], and Larsen [Lar09] consider misspecifications of the utility functions. Market perturbations are considered in Larsen and Žitković [LŽ07] -who, working in a continuous time market, investigate the continuous dependence of the value function and of

the agent's optimal behavior on the price of the stock (parametrized by the market-price-of-risk), and in Kardaras and Žitković [KŽ11], who perform a stability analysis of the problem under small misspecifications of (the agent's preferences and of) the market model as modeled via subjective probabilities. Hubalek and Schachermayer [HS98] study the convergence of prices of illiquid assets when the prices of the liquid assets converge, and stability of option pricing under market perturbations has been investigated by El Karoui et al. [EKJPS98], while Kardaras [Kar10] looks at the stability of the numéraire portfolio¹.

The previously mentioned results only deal with stability, and constitute a zeroth order approach to the problem. References which perform a first-order study are Henderson [Hen02] -who studies, in a Brownian market, the expansion of the indifference price with respect to a small number of random endowments (see also Henderson and Hobson [HH02])- and Kramkov and Sirbu [KS06] -who generalize the first order approximation to semimartingale markets.

A different setting, which has been popular in recent years, is the one where investors can trade continuously in time in a number of assets, as well as take static positions in some derivatives; for examples of papers that consider this framework, see Campi [Ca04, Ca11], Ilhan et. al [IS06, IJS05] and Carr et al. [CJM01], Schweizer and Wissel [SW08a, SW08b], Jacod and Protter [JP10], [Sio13].

In particular Ilhan et al. [IJS05] work in a market where *some* of the contingent claims can be traded only at time zero, and so their prices are modeled simply as a vector in \mathbb{R}^n , a much simpler object than a general \mathbb{R}^n -valued semimartingale. In this simplified setting and using an exponential utility, they are able to (study existence and uniqueness of the solution and to) obtain *differentiability and strictly-convexity* of the value function *as a function of the price* of the financial derivatives.

In the present paper we work in the same semi-static framework as Ilhan et al. [IJS05] -assuming that some assets can be traded continuously in time, while others can be traded only at time zero (we will think of the liquid part of the market as being composed of stocks, and the illiquid one by derivatives)- but we consider a general utility function defined on the positive real line. In this setting, we study the dependence of the value function, of its maximizer and of other quantities of interest on the (initial capital and on the) price p of the derivatives; thanks to the nature of our framework, we are able to prove not only stability, but also differentiability, monotonicity, convexity and asymptotic properties. Specifically, we are able to reproduce in our model a number of economically sensible properties: the value function u has the expected monotonic behavior (as p increases u is at first decreasing, then constant, then increasing), and it diverges (along with the optimizer) when the price of the derivatives converge to an arbitrage price. For a study of existence and uniqueness of the optimizer of this utility maximization problem we refer to Siorpaes [Sio13].

¹Which is defined as the numéraire under which all other wealth processes become supermartingales under the historical measure; when the log-optimal portfolio exists, it coincides with the numéraire portfolio.

We also show convexity in p of the largest feasible position, defined as the maximum number of shares of derivatives that the agent with given initial wealth can buy at price p and still be able to invest in the (liquid) stock market as to have a non-negative final wealth. This simple fact, which is a consequence of the no-arbitrage assumption, is perhaps somewhat surprising (especially when there are multiple derivatives) and -to the best of our knowledge- was not noticed before. Unlike in the case of exponential utility, we show with an example that the maximal expected utility does not need to be a convex function of the derivative's prices; this makes proving differentiability a much trickier task, which we are able to carry on only under additional assumptions (most importantly in the case of power utility).

On the technical side, we emphasize that this optimization problem does not follow under the general umbrella of utility maximization with convex constraints (see Larsen and Žitković [LŽ] for a survey), since it cannot be re-phrased asking that the portfolio and wealth process lie in some given convex set (possibly depending on t and ω); rather, we demand that the investor, after choosing his position at time zero *arbitrarily*, keep his position in derivatives unchanged for the rest of the time horizon, while freely investing in stocks. Moreover, considering an exponential utility allows Ilhan et al. [IJS05] to make essential use of explicit representations of the maximal utility and of indifference prices; however, as they point out, the 'extension to more general cases is not trivial', as a different approach is needed.

A simple but crucial observation that we will rely on is that our problem can be decomposed in two steps: choosing the optimal amount of derivatives to buy at time zero, and then investing optimally in the continuous time stock market; the second step being the problem of optimal investment with random endowment, which is then closely related to our problem. Alternating this point of view with the one where the whole optimization problem is embedded in the abstract framework of Kramkov and Schachermayer [KS99],[KS03] allows us to deduce the continuity properties via convex duality.

The paper is organized as follows. In Section 1 we succinctly present the model of financial market and we define our problem. In Section 2 we state our main theorems. In Section 3 we establish the continuity of the outputs of the utility maximization problem. In Section 4 we prove the convexity of the largest feasible position and we study the asymptotic behavior of the value function and the optimizer. In Section 5 we consider in more detail the one-dimensional case, and in Section 6 we provide an example of a value function which is not convex in p . Finally, in Section 7 we study the differentiability of the value function.

1 The model

We work in the same framework as in [Sio13], and refer to this paper for more details and references. We consider at first a model of a financial market composed of a savings account and d stocks which can be traded continuously in time. We consider

a finite deterministic² time horizon $[0, T]$, and we assume that the interest rate is 0. The price process $S = (S^i)_{i=1}^d$ of the stocks is assumed to be a locally-bounded³ semimartingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfies the usual conditions.

Now enlarge the market by allowing also n contingent claims $f = (f_j)_{j=1}^n$ to be traded at price $p = (p_j)_{j=1}^n$; we assume that these contingent claims can be traded *only* at time zero, and that p is an arbitrage-free price for the European contingent claims f (in a sense which will be specified later).

A self-financing portfolio is then defined as a triple (x, q, H) , where $x \in \mathbb{R}$ represents the initial capital, $q_j \in \mathbb{R}$ represents the holding in the contingent claim f_j , and the random variable H_t^i specifies the number of shares of stock i held in the portfolio at time t .

An agent with portfolio (x, q, H) will invest his initial wealth x buying q European contingent claims at price p at time zero. This quantity is then held constant up to maturity, so the vector q represents the illiquid part of the portfolio and $qp := \sum_{j=1}^n q_j p_j$ represents⁴ the wealth invested in the European contingent claims.

He will then invest the remaining wealth $x - qp$ dynamically, buying H_t share of stocks at time $t \in [0, T]$, and put the rest (positive or negative) into the savings account. We will denote by X_t the wealth process, which denotes the value of the dynamic part of the portfolio, and which evolves in time as the stochastic integral of H with respect to S :

$$X_t = x - qp + (H \cdot S)_t = x - qp + \int_0^t H_u dS_u, \quad t \in [0, T],$$

where H is assumed to be a predictable S -integrable process.

For $x \geq 0$, we denote by $\mathcal{X}(x)$ the set of non-negative wealth processes whose initial value is equal to x , that is,

$$\mathcal{X}(x) := \{X \geq 0 : X_t = x + (H \cdot S)_t\}.$$

A probability measure Q is called an *equivalent local-martingale measure* if it is equivalent to P , and if S is a local-martingale under Q . We denote by \mathcal{M} the family of equivalent local-martingale measures, and we assume that

$$\mathcal{M} \neq \emptyset. \tag{1}$$

In our model we consider an agent whose preferences are modeled via a utility function $U : (0, \infty) \rightarrow \mathbb{R}$, which is assumed to be strictly concave, strictly increasing

²For simplicity, the time horizon $T > 0$ is assumed to be constant; however, T could be replaced by a finite stopping time, as is the case in Hugonnier and Kramkov (2004), on which we rely.

³This assumption is not really necessary, as the results in [KS99], [KS03], [HK04], [DS97] on which our proofs hinge, although proved for a locally-bounded semimartingale, are true also without the local boundedness assumption, as long as one replaces equivalent local-martingale measures with separating measures throughout. This fact is stated in [JHS05, Remark 3.4].

⁴In this paper vw will always denote the dot product between two vectors v and w , and $|\cdot|$ will denote the Euclidean norm in \mathbb{R}^{n+1} .

and continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x \rightarrow 0+} U(x) = \infty, \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0. \quad (2)$$

We extend the utility U to the whole real line by setting $U(x)$ to be $-\infty$ for x in $(-\infty, 0)$, and to equal $U(0+) := \lim_{x \rightarrow 0} U(x)$ at $x = 0$.

We denote by $f = (f_j)_{j=1}^n$ the family of the \mathcal{F}_T -measurable payment functions of the European contingent claims with maturity T , and by $qf = \sum_{j=1}^n q_j f_j$ the payoff of the static part of the portfolio. The total payoff of the portfolio (x, q, H) is then $x - qp + (H \cdot S)_T + qf$.

A non-negative wealth process in $\mathcal{X}(x)$ is said to be maximal if its terminal value cannot be dominated by that of any other process in $\mathcal{X}(x)$. We assume that the European contingent claims can be sub- and super-replicated by trading in the stock; in other words, that there exists a non-negative maximal wealth process X' such that

$$|f| := \sqrt{\sum_{j=1}^n f_j^2} \leq X'_T. \quad (3)$$

A strictly positive maximal wealth process is called a numéraire, and a wealth process X is called *acceptable* if it is bounded below under some numéraire (i.e., $X/N \geq c$ for some numéraire N and constant c). Since condition (3) is equivalent to saying that $|f|$ is bounded with respect to some numéraire, the acceptable processes constitute a natural optimization set for our optimal investment problem.

Following [HK04], we define $\mathcal{X}(x, q)$ to be the set of acceptable wealth processes with initial value x whose terminal value dominates the random payoff $-qf$, i.e.,

$$\mathcal{X}(x, q) := \{X : X \text{ is acceptable, } X_0 = x \text{ and } X_T + qf \geq 0\}.$$

We will call $\bar{\mathcal{K}}$ the set of points (x, q) where $\mathcal{X}(x, q)$ is not empty, i.e.,

$$\bar{\mathcal{K}} := \{(x, q) \in \mathbb{R} \times \mathbb{R}^n : \mathcal{X}(x, q) \neq \emptyset\}. \quad (4)$$

As shown in [HK04, Lemma 6], assumptions (1) and (3) imply that the convex cone $\bar{\mathcal{K}}$ defined in (4) is closed and its interior \mathcal{K} contains $(x, 0)$ for any $x > 0$, so $\bar{\mathcal{K}}$ is the closure of \mathcal{K} .

We will say that p is an *arbitrage-free price* for the European contingent claims f if any portfolio with zero initial capital and non-negative final wealth has identically zero final wealth; in other words p is an arbitrage-free price if, $\forall q \in \mathbb{R}^n$, $X \in \mathcal{X}(-pq, q)$ implies $X_T = -qf$. We will denote by \mathcal{P} the set of arbitrage-free prices.

The objective of this paper is to study the price dependence in the problem of utility maximization in the enlarged market consisting of the bond, the stocks and the contingent claims, i.e., the following optimization problem⁵, for $x > 0, p \in \mathcal{P}$,

$$\tilde{u}(x, p) := \sup \{\mathbb{E}[U(X_T + qf)] : X \text{ is acceptable, } q \in \mathbb{R}^n, X_0 = x - qp\}. \quad (5)$$

⁵By convention, we set $\mathbb{E}[U(X_T + qf)]$ equal to $-\infty$ when $\mathbb{E}[U^-(X_T + qf)] = -\infty$ (whether or not $\mathbb{E}[U^+(X_T + qf)]$ is finite).

The problem of utility maximization in presence of a random endowment^{3,6}

$$u(x, q) := \sup_{X \in \mathcal{X}(x, q)} \mathbb{E}[U(X_T + qf)], \quad (x, q) \in \mathbb{R}^{n+1}, \quad (6)$$

is obviously closely related to (5); in fact it is easy to show that $\mathcal{K} \subseteq \{u > -\infty\} \subseteq \bar{\mathcal{K}}$ (see [Sio13, Theorem 8]), and trivially

$$\tilde{u}(x, p) = \sup_{q \in \mathbb{R}^n} u(x - qp, q) = \sup_{q \in \mathbb{R}^n : (x - qp, q) \in \bar{\mathcal{K}}} u(x - qp, q), \quad x > 0, p \in \mathcal{P}. \quad (7)$$

Existence and uniqueness of the solution (\tilde{q}, \tilde{X}_T) of (5) has been established in [Sio13], where also the relationship with problem (6) has been investigated.

In this paper we study the dependence on the initial price p of the outputs of problem (5): the optimal position in derivatives \tilde{q} , the optimal final value of the dynamic part of the portfolio \tilde{X}_T , the maximal expected utility \tilde{u} (and its derivatives), and the *largest feasible position* $m : (0, \infty) \times \mathcal{P} \rightarrow [0, \infty]$, defined as

$$m(x, p) := \sup \{ |q| : q \in \mathbb{R}^n, (x - qp, q) \in \bar{\mathcal{K}} \}, \quad (8)$$

which measures the maximum number of shares of derivatives that the agent with wealth x can buy at price p and still be able to invest in the stock market as to have a non-negative final wealth.

2 Statement of the main theorems

To state the main theorems we need to introduce some notation. Denote by $\mathcal{Y}(y)$ the family of non-negative processes Y with initial value y and such that for any non-negative wealth process X the product XY is a super-martingale, that is,

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y, XY \text{ is a super-martingale for all } X \in \mathcal{X}(1)\}.$$

In particular, as $\mathcal{X}(1)$ contains the constant process 1, the elements of $\mathcal{Y}(y)$ are non-negative super-martingales. Note also that the set $\mathcal{Y}(1)$ contains the density processes of all $Q \in \mathcal{M}$.

The convex conjugate function V of the agent's utility function U is defined to be the Fenchel-Legendre transform of the function $-U(\cdot)$; that is,

$$V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy) = \sup_{x > 0} (U(x) - xy), \quad y \in \mathbb{R}.$$

It is well known that, under the Inada conditions (2), the conjugate of U is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(0) = -\infty$, $V'(\infty) = 0$ and $V(0) = U(\infty)$, $V(\infty) = U(0)$ as well as the following bi-dual relation:

$$U(x) = \inf_{y \in \mathbb{R}} (V(y) + xy) = \inf_{y > 0} (V(y) + xy), \quad x \in \mathbb{R}.$$

⁶We use the convention that the sup (inf) over an empty set takes the value $-\infty$ ($+\infty$).

Following [HK04], we will denote by w the value function of the problem of optimal investment without the European contingent claims, and by \tilde{w} its dual value function. In other words

$$w(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0; \quad \tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0.$$

Recall that a random variable g is said to be *replicable* if there is an acceptable process X such that $-X$ is also acceptable and $X_T = g$ (if such a process exists, it is unique). In order to have uniqueness of the maximizer of (7) we will assume that

$$\text{for any non-zero } q \in \mathbb{R}^n \text{ the random variable } qf \text{ is not replicable.} \quad (9)$$

Note that, by discarding some contingent claims, one can always reduce to the case where (9) holds (see [HK04, Remark 6]), so assuming (9) does not comport a loss of generality.

The following is our first main theorem; it does not have an analogue in [IJS05] and, we believe, is very intuitive (given its economic interpretation). It shows that the dependence on p takes a particularly pleasing form in the case where q is scalar-valued.

Theorem 1 *Assume that p is an arbitrage-free price for f , that conditions (1), (2), (3) and (9) hold true, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0.$$

If there is only one European contingent claim f , for any $x > 0$ there exists $\underline{p} < a \leq b < \bar{p}$ such that $\mathcal{P}(x, 0) = [a, b] \subset (\underline{p}, \bar{p}) = \mathcal{P}$. The function

$$\begin{aligned} (\underline{p}, \bar{p}) &\longrightarrow \mathbb{R} \\ p &\mapsto \tilde{q}(x, p) \end{aligned}$$

is continuous, it is strictly positive on (\underline{p}, a) , it equals zero on $[a, b]$, and it is strictly negative on (b, \bar{p}) . The function

$$\begin{aligned} (\underline{p}, \bar{p}) &\longrightarrow \mathbb{R} \\ p &\mapsto \tilde{u}(x, p) \end{aligned}$$

is continuous, it is strictly decreasing on (\underline{p}, a) , it is constant on $[a, b]$, and it is strictly increasing on (b, \bar{p}) .

Moreover if $U(\infty) = \infty$ then $-\tilde{q}(x, \bar{p}-) = \tilde{q}(x, \underline{p}+) = \tilde{u}(x, \underline{p}+) = \tilde{u}(x, \bar{p}-) = \infty$.

We recall that, in the general setting in which we work, $\mathcal{P}(x, 0)$ is not a singleton (see [JHS05, Theorem 3.1]), so in the above theorem it could be that $a < b$.

Our second main theorem, which deals with the general multi-dimensional setting, shows that the problem of optimal investment is well posed, and that, when the arbitrage-free prices converge to an arbitrage price, the corresponding utility and the optimal demand diverge; moreover, we show that m is a convex function of the derivative's price. In regards to the continuity, we can actually obtain a more detailed statement: see Theorem 4.

Theorem 2 *Under the assumptions of Theorem 1, \mathcal{P} is an open bounded convex set, the map $m(x, p)$ is finite valued, the map*

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow \mathbb{R}^n \times L^0(P) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ (x, p) &\mapsto \left(\tilde{q}, \tilde{X}_T, \tilde{u}, \partial_x \tilde{u}, m \right) \end{aligned}$$

is continuous, and $m(x, p)$ is positively homogeneous in x , convex in p , and the supremum in (8) is attained. Moreover if $U(\infty) = \infty$ and $\mathcal{P} \ni p_n \rightarrow p \notin \mathcal{P}, x_n \rightarrow x > 0$ then

$$\tilde{u}(x_n, p_n) \rightarrow \infty = \tilde{u}(x, p), \quad m(x_n, p_n) \rightarrow \infty \text{ and } |\tilde{q}(x_n, p_n)| \rightarrow \infty$$

as $n \rightarrow \infty$.

We remark that in [IJS05] the function $p \mapsto \tilde{u}(x, p)$ (corresponding to an exponential utility U) is proved to be strictly convex and differentiable. However, we will show that in our general framework convexity does not hold; this, and the fact that the maximizer of (7) may lay on the boundary of \mathcal{K} (see [Sio12, Section 4]) make proving differentiability in p a very delicate task. The next theorem shows that these are the only impediments to differentiability, and that they can be circumvented in some occasions.

Theorem 3 *Under the assumptions of Theorem 1, the function*

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow \mathbb{R} \\ (x, p) &\mapsto \tilde{u}(x, p) \end{aligned}$$

is continuously differentiable at all points if the function U is a power utility.

In general, \tilde{u} is continuously differentiable in a neighborhood of some point (x^, p^*) if either of the following conditions is satisfied:*

1. *The optimizer $(x^* - q^* p^*, q^*)$ belongs to \mathcal{K} , where $q^* := \tilde{q}(x^*, p^*)$.*
2. *The function $p \mapsto \tilde{u}(x^*, p)$ is convex in a neighborhood of p^* .*

Moreover, whenever the derivatives exist, they satisfy $\nabla_p \tilde{u} = -(\partial_x \tilde{u}) \tilde{q}$.

To show that the optimizer $(x^* - q^* p^*, q^*)$ belongs to \mathcal{K} one can use the fact that it equals $-\nabla \tilde{u}(\partial_x \tilde{u}, p \partial_x \tilde{u})$ (see [Sio13, Theorem 2]), which can occasionally be computed explicitly.

We remark that the equation $\nabla_p \tilde{u} = -\tilde{q} \partial_x \tilde{u}$ holds also for exponential utilities (see [IJS05, Theorem 3.1]⁷), and that it can be derived heuristically⁸ in a simple

⁷Actually, [IJS05, Theorem 3.1] states that $\nabla_p \tilde{u} = \tilde{q}$; the missing minus sign in front of \tilde{q} (which they call λ^*) is a typo, whereas the term $\partial_x \tilde{u}$ is missing because in their case $\partial_x \tilde{u} = 1$ (this follows from [IJS05, Theorem 4.1]).

⁸This is not a complete proof, as it is unclear at which points (x', p) the equality is satisfied.

fashion. Indeed, p is a marginal price at (x, q) iff the agent with endowment (x, q) could achieve no gains by trading in derivatives; in other words we obtain

$$p \in \mathcal{P}(x, q) \quad \text{iff} \quad \tilde{u}(x + qp, p) = u(x, q) = \min_{p'} \tilde{u}(x + qp', p'), \quad (10)$$

which is somewhat interesting, as it allows us to characterize marginal prices using \tilde{u} instead of u ; moreover, note how the last equality is in some sense dual to (7). Anyway, it follows from (10) that the function $g(p') = \tilde{u}(x + qp', p')$ has gradient zero at $p' = p$, i.e., $0 = (q\partial_x \tilde{u} + \nabla_p \tilde{u})(x + qp, p)$. Define $x' := x + qp$, then $p \in \mathcal{P}(x, q)$ implies $q = \tilde{q}(x', p)$, and so $0 = (\tilde{q}\partial_x \tilde{u} + \nabla_p \tilde{u})(x', p)$.

3 Continuity

We first need to introduce the dual problems. If we define the set $\bar{\mathcal{L}}$ to be the polar of $-\bar{\mathcal{K}}$:

$$\bar{\mathcal{L}} := -\bar{\mathcal{K}}^\circ := \{v \in \mathbb{R}^{n+1} : vw \geq 0 \text{ for all } w \in \bar{\mathcal{K}}\},$$

then clearly $\bar{\mathcal{L}}$ is a closed convex cone. We will denote by \mathcal{L} its relative interior, so $\bar{\mathcal{L}}$ is the closure of \mathcal{L} . Given an arbitrary vector $(y, r) \in \mathbb{R} \times \mathbb{R}^n$, we denote by $\mathcal{Y}(y, r)$ the set of non-negative super-martingales $Y \in \mathcal{Y}(y)$ such that the inequality

$$\mathbb{E}[Y_T(X_T + qf)] \leq xy + qr$$

holds whenever $(x, q) \in \bar{\mathcal{K}}$ and $X \in \mathcal{X}(x, q)$; it is easy to show that $\mathcal{Y}(y, r)$ is non-empty if and only if $(y, r) \in \bar{\mathcal{L}}$ (see [Sio12, Remark 5]).

We now define the problems dual to (5) and to (6) to as follows⁴:

$$\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y, yp)} \mathbb{E}[V(Y_T)], \quad y \in \mathbb{R} \quad (11)$$

and

$$v(y, r) = \inf_{Y \in \mathcal{Y}(y, r)} \mathbb{E}[V(Y_T)], \quad (y, r) \in \mathbb{R} \times \mathbb{R}^n, \quad (12)$$

where p is the vector of prices of the contingent claims f . We recall that the relationship between w, \tilde{w} and u, v is given by $u(x, 0) = w(x)$ and $\tilde{w}(y) = \min_{p \in \mathcal{P}} v(y, yp)$ ⁹, and that $\mathcal{L} = \{(y, yp) : y > 0 \text{ and } p \in \mathcal{P}\}$ (see [Sio12, Lemma 3]) and $(x - \tilde{q}p, \tilde{q}) = -\nabla v(\tilde{y}, \tilde{y}p)$, where $\tilde{y} = \tilde{u}'(x)$ (see [Sio13, Theorem 2]).

Find here more detailed continuity results than those announced in Theorem 2.

Theorem 4 *Assume that p is an arbitrage-free price for f , that conditions (1), (2), (3) and hold, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0.$$

⁹The first one follows from $\mathcal{X}(x, 0) = \mathcal{X}(x)$, the second one is the content of [HK04, Lemma 2]

Then the maps

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow L^0(P) \times L^1(P) \times \mathbb{R} \times \mathbb{R} \\ (x, p) &\mapsto \left(\tilde{X}_T + \tilde{q}f, U(\tilde{X}_T + \tilde{q}f), \tilde{u}(x, p), \frac{\partial \tilde{u}(x, p)}{\partial x} \right) \end{aligned}$$

and

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow L^0(P) \times L^1(P) \times \mathbb{R} \times \mathbb{R} \\ (y, p) &\mapsto \left(\tilde{Y}_T, V(\tilde{Y}_T), \tilde{v}(y, p), \frac{\partial \tilde{v}(y, p)}{\partial y} \right) \end{aligned}$$

are continuous, and if we additionally assume (9) then the map

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow \mathbb{R}^n \times L^0(P) \\ (x, p) &\mapsto (\tilde{q}, \tilde{X}_T) \end{aligned}$$

is (well defined and) continuous.

PROOF: Step 1 To obtain the proof for $(\tilde{Y}_T, V(\tilde{Y}_T))$ apply the changes which we will now describe to the proof of [KS99, Lemma 3.6]. Replace y and y_n with (y, yp) and $(y_n, y_n p_n)$, \hat{h} with \tilde{Y}_T and the function that is there called v (which in this article we denote by \tilde{w}) with the function v defined in (12). Apply [HK04, Lemma 2] to obtain the finiteness of v from our assumption that \tilde{w} is finite, and then use [Sio12, Lemma 4] to obtain $g \in \mathcal{D}(y, yp)$. Finally use $\mathcal{D}(y, yp) \subseteq \mathcal{D}(y)$ and the fact that v , being concave and finite on the open set $\mathcal{L} \ni (y, yp)$, is there continuous; this concludes the proof that for $(\tilde{Y}_T, V(\tilde{Y}_T))$ is continuous. The continuity of \tilde{v} is now trivial, and the continuity of its derivative in y follows from [Roc70, Theorem 25.7].

Step 2 Since $\tilde{u}(\cdot, p)$ is concave, it is locally Lipschitz, so $\tilde{u}(\cdot, \cdot)$ is continuous if $\tilde{u}(x, \cdot)$ is continuous. Since $\tilde{u}(x, \cdot)$ is the infimum over $y > 0$ of the continuous functions $p \mapsto v(y, yp) + xy$, it is upper-semi-continuous. To prove its lower semi-continuity define the open set

$$V_p := \{q \in \mathbb{R}^n : (x - qp, q) \in \mathcal{K}\},$$

and observe that, since u is upper-semi-continuous (see [Sio12, Theorem 6]) and concave, [Roc70, Theorem 7.5] yields

$$u(a, b) = \lim_{\varepsilon \rightarrow 0+} u((a, b)(1 - \varepsilon) + (1, 0)\varepsilon) \text{ for all } (a, b) \in \bar{\mathcal{K}},$$

which implies that for any $c \in \mathbb{R}$

$$\tilde{u}(x, p) \leq c \text{ if and only if } u(x - qp, q) \leq c \text{ for all } q \in V_p. \quad (13)$$

Since u is continuous when restricted to \mathcal{K} , if $p_n \rightarrow p \in \mathcal{P}$ and $\tilde{u}(x, p_n) \leq c$ then for all $q \in V_p$

$$u(x - qp, q) = \lim_n u(x - qp_n, q) \leq c,$$

so (13) implies $\tilde{u}(x, p) \leq c$, so $\{\tilde{u}(x, \cdot) \leq c\}$ is closed, i.e., $\tilde{u}(x, \cdot)$ is lower-continuous and so $\tilde{u}(\cdot, \cdot)$ is continuous¹⁰. The continuity of $\partial_x \tilde{u}$ now follows from [Roc70, Theorem 25.7]. As proved in [Sio13, Theorem 2], \tilde{q} is uniquely defined iff (9) holds, and in this case $\tilde{q} = -\nabla v(\partial_x \tilde{u}, \partial_x \tilde{u} p)$, so \tilde{q} is continuous.

Step 3 Now we describe the changes one needs to apply to the proof of [KS99, Lemma 3.6] in order to obtain continuity of $(\tilde{X}_T + \tilde{q}f, U(\tilde{X}_T + \tilde{q}f))$. Replace y and y_n with (x, p) and (x_n, p_n) , \hat{h} with $\tilde{X}_T + \tilde{q}f$, V with $-U$ and the function that is there called v with the function \tilde{u} . Observe that we can assume without loss of generality that (9) holds, so if $\tilde{q}_n := \tilde{q}(x_n, p_n)$ and $\tilde{q} := \tilde{q}(x, p)$ then $\tilde{q}_n \rightarrow \tilde{q}$. Define

$$h_n := \frac{(\tilde{X}_T(x_n, p_n) + \tilde{X}_T(x, p)) + (\tilde{q}(x_n, p_n) + \tilde{q}(x, p))f}{2},$$

so that $h_n \in \mathcal{C}(a_n, b_n)$, where

$$(a_n, b_n) := \left(\frac{x_n + x}{2} - \frac{(\tilde{q}_n p_n + \tilde{q}p)}{2}, \frac{\tilde{q}_n + \tilde{q}}{2} \right).$$

Since (a_n, b_n) is a convergent sequence, (3) provides an $\bar{x} > 0$ such that $\mathcal{C}(a_n, b_n)$ is contained in $\mathcal{C}(\bar{x}, 0)$, which is bounded in $L^0(P)$. This yields [KS99, Formula (3.13)] in our case, and also allows to apply Kolmos' lemma to construct a sequence $(g_n)_{n \geq 1}$ of forward convex combinations of $(h_n)_{n \geq 1}$ that is converging almost surely to some random variable g , which [Sio12, Lemma 4] shows to be in $\tilde{\mathcal{C}}(x, p)$. Since $(g_n)_{n \geq 1} \subseteq \mathcal{C}(\bar{x}, 0)$, we can apply [KS03], lemma 1 to prove the uniform integrability of $U^+(g_n)$, using the continuity of \tilde{u} . \square

4 Asymptotic results and convexity of m

In this section we will prove in steps Theorem 2.

Lemma 5 *If (1), (3) hold, \mathcal{P} is bounded and convex, and it is open iff (9) holds.*

PROOF. The identity $\mathcal{P} = \{p : (1, p) \in \mathcal{L}\}$ shows that \mathcal{P} is convex, and is open if and only if condition (9) is satisfied (see [HK04, Lemma 3]). Assume by contradiction that \mathcal{P} is not bounded, i.e., there exists $p_n \in \mathcal{P}$ such that $|p_n|$ is converging to infinity as $n \rightarrow \infty$, and define

$$q_n := -p_n/|p_n|^{3/2}. \quad \text{Then } 1 + q_n p_n = 1 - |p_n|^{1/2}$$

is converging to $-\infty$ and so it is negative for big enough n . Since by the bipolar theorem $\bar{\mathcal{K}}$ is the polar of $\tilde{\mathcal{L}} \ni (1, p_n)$, this implies that $(1, q_n) \notin \bar{\mathcal{K}}$, which by [HK04,

¹⁰We can also give a much more elegant proof that $\tilde{u}(x, \cdot)$ is continuous, relying on a hard-to-prove theorem: since \tilde{v} is continuous and $\tilde{u}(x) = \tilde{v}(\tilde{y}) + x\tilde{y}$, where $\tilde{y} = \partial_x \tilde{u}$, the continuity of \tilde{u} follows from the one of \tilde{y} . To prove the latter, observe that the continuous bijection g of $(0, \infty) \times \mathcal{P}$ in itself given by $(y, p) \mapsto (-\partial_y \tilde{v}(y, p), p)$ has inverse $g^{-1}(x, p) = (\tilde{y}, p)$, and the map g is open, by Brouwer's invariance domain theorem, so g^{-1} is continuous.

Lemma 1] contradicts (3), since $q_n \rightarrow 0$. \square

To obtain our results on the function m , we will first study the following auxiliary function $d : \mathcal{L} \rightarrow [0, \infty]$ defined as

$$d(w) := \sup\{|v| : v \in \bar{\mathcal{K}} \text{ and } vw \leq 1\}. \quad (14)$$

Lemma 6 *Assume conditions (1), (3), (9), then*

1. $xd(w) = d(\frac{w}{x})$ for every $x > 0, w \in \mathcal{L}$.
2. d is finite and the supremum in (14) is attained.
3. $d(w_1 + w_2) \leq \min(d(w_1), d(w_2))$ for all $w_1, w_2 \in \mathcal{L}$.
4. d is locally bounded.

PROOF. Item one is obvious, and then [Sio12, Lemma 3] implies that d is finite at every point, so clearly by compactness the supremum defining d is attained. Item three follows from $\bar{\mathcal{L}} = -\bar{\mathcal{K}}^\circ$. Finally, let $(h_i)_{i=1}^k \subset \mathcal{L}$ be the vertices of an open cube Q containing w (where clearly $k = 2^{n+1}$), and $\lambda = (\lambda_i)_{i=1}^k$ be convex weights. Let $h_\lambda := \sum_{i=1}^k \lambda_i h_i$ be the generic point in Q , and choose j such that $\lambda_j \geq 1/k$; using item one and three we obtain

$$d(h_\lambda) \leq d(\lambda_j h_j) \leq kd(h_j) \leq k \max_{1 \leq i \leq k} d(h_i) =: C_Q < \infty.$$

Thus d is bounded on Q , so d is locally bounded. \square

PROOF OF THEOREM 2. The continuity properties follow from Theorem 4, and the properties of \mathcal{P} from Lemma 5; let us prove the asymptotic properties, assuming $U(\infty) = \infty$. Since $\tilde{u}(\cdot, p)$ is increasing we can assume without loss of generality that $x_n = x$. Since p is not an arbitrage-free price, by definition there exists an $X \in \mathcal{X}(-qp, q)$ such that the inequality $X_T \geq 0$ holds and is strict with strictly positive probability. By monotone convergence

$$\lim_n \mathbb{E}[U(x + nX_T)] = P(X_T = 0)U(x) + P(X_T > 0)U(\infty) = \infty,$$

and so $\tilde{u}(x, p) = \infty$ for any $x > 0$, and given an arbitrary $M > 0$ we can find a $q \in \mathbb{R}^n$ such that $u(\frac{x}{2} - qp, q) \geq M$. But then for n big enough $q(p_n - p) < \frac{x}{2}$ and so

$$\tilde{u}(x, p_n) \geq \tilde{u}(\frac{x}{2} + q(p_n - p), p_n) \geq u(\frac{x}{2} + q(p_n - p) - qp_n, q) = u(\frac{x}{2} - qp, q) \geq M,$$

which proves $\lim_n \tilde{u}(x, p_n) = \infty$.

If the sequence $q_n := \tilde{q}(x_n, p_n)$ was converging to some $q \in \mathbb{R}^n$ the upper-semicontinuity of u would imply

$$\infty = \lim_n \tilde{u}(x, p_n) = \lim_n u(x - q_n p_n, q_n) \leq u(x - qp, q),$$

which is not possible since the concave function u is real valued on some open set. It follows analogously that q_n can not have any convergent subsequence, which implies $\lim_n |\tilde{q}(x_n, p_n)| = \infty$; since $|\tilde{q}| \leq m$, also $m(x_n, p_n)$ diverges. Let us now prove the results about the function m . The positive homogeneity in x is trivial and, since $\mathcal{P} = \{p : (1, p) \in \mathcal{L}\}$ (see [Sio12, Lemma 3]), $m(1, p)$ is bounded above by $d(w)$ for $w := (1, p)$. Thus m is finite, and by compactness this implies that the supremum in (8) is attained. To conclude, we only need to show that $m(1, \cdot)$ is convex, as this will also imply continuity of m . Note that

$$m(x, p) = \max \left\{ |q| : (z, q) \in (\mathbb{R} \times \mathbb{R}^n) \cap \bar{\mathcal{K}} \text{ and } (z, q)(1, p) \leq x \right\}, \quad (15)$$

since trivially any maximizer (\bar{z}, \bar{q}) satisfies $(\bar{z}, \bar{q})(1, p) = x$. From Lemma 6 it follows that, for fixed $w \in \mathcal{L}$, there exists $\varepsilon > 0$ s.t.

$$2\varepsilon \sup \left\{ d(w') : w' \in B_\varepsilon(w) \right\} < 1, \quad (16)$$

where $B_\varepsilon(w)$ is the ball of radius ε centered at w . Let $p \in \mathcal{P}$, $p_0, p_1 \in B_\varepsilon(p)$, and define $w := (1, p)$ and $w_i := (1, p_i)$ for $i = 0, 1$. Fix a generic $\lambda \in (0, 1)$, let

$$p_\lambda := \lambda p_1 + (1 - \lambda)p_0 \text{ and } w_\lambda := \lambda w_1 + (1 - \lambda)w_0 = (1, p_\lambda),$$

and use (15) to choose a $v_\lambda = (z_\lambda, q_\lambda) \in (\mathbb{R} \times \mathbb{R}^n) \cap \bar{\mathcal{K}}$ that satisfies $|q_\lambda| = m(1, p_\lambda)$ and $v_\lambda w_\lambda \leq 1$, so that necessarily $v_\lambda w_\lambda = 1$. Suppose that we can build $t_0, t_1 \in \mathbb{R}$ such that

$$t_i > 0, \quad t_i v_\lambda w_i \leq 1 \text{ for } i = 0, 1 \text{ and } 1 = \lambda t_1 + (1 - \lambda)t_0. \quad (17)$$

Then, if we define $v_i := (z_i, q_i) := t_i(z_\lambda, q_\lambda) = t_i v_\lambda$ for $i = 0, 1$, we have that $v_i \in \bar{\mathcal{K}}$, $v_i w_i \leq 1$ and $v_\lambda := \lambda v_1 + (1 - \lambda)v_0$, and so (15) implies

$$m(1, p_\lambda) = |q_\lambda| \leq \lambda |q_1| + (1 - \lambda)|q_0| \leq \lambda m(1, p_1) + (1 - \lambda)m(1, p_0),$$

which shows that $m(1, \cdot)$ is locally convex and thus¹¹ convex; so, to conclude we just need to build t_0, t_1 that satisfy (17). To do so, define t_1 such that $t_1 v_\lambda w_1 = 1$, i.e.,

$$t_1 := \frac{1}{1 + (1 - \lambda)v_\lambda(w_1 - w_0)}.$$

Note that t_1 is (defined and) strictly positive, since (16) implies that

$$|v_\lambda(w_1 - w_0)| \leq 2|v_\lambda|(w_1 - w_0) \leq 2d(1, w_\lambda)\varepsilon < 1. \quad (18)$$

Now, define t_0 such that $1 = \lambda t_1 + (1 - \lambda)t_0$ holds. It is easy to show that (18) implies $t_0 > 0$, and somewhat lengthy but straightforward¹² algebraic manipulations show that $t_0 v_\lambda w_0 \leq 1$ is equivalent to $(\lambda - 1)(v_\lambda(w_1 - w_0))^2 \leq 0$, and so it is satisfied. \square

¹¹It is enough to show this in dimension one, where a function is convex iff it is the integral of an increasing function; since a locally increasing functions is increasing, the thesis follows.

¹²Just remember to use the identities $w_0 = w_\lambda - \lambda(w_1 - w_0)$ and $v_\lambda w_\lambda = 1$.

5 The one dimensional case

To prove Theorem 1 we will need the following lemma.

Lemma 7 *Under the assumptions of Theorem 1, if there is only one European contingent claim f , and $p_1, p_2 \in \mathcal{P}, x > 0$, then*

1. *If $p_2 < p_1$ and $\tilde{q}(x, p_1) > 0$ then $\tilde{q}(x, p_2) > 0$ and $\tilde{u}(x, p_1) < \tilde{u}(x, p_2)$.*
2. *If $p_2 > p_1$ and $\tilde{q}(x, p_1) < 0$ then $\tilde{q}(x, p_2) < 0$ and $\tilde{u}(x, p_1) < \tilde{u}(x, p_2)$.*

PROOF Assume that $p_2 < p_1$ and $q_1 := \tilde{q}(x, p_1) > 0$, and let $a_1 := x - q_1 p_1$, then $(a_1, q_1)(1, p_2) = x - q_1(p_1 - p_2) < x$, and so there exists $t_1 > 1$ s.t. $t_1(a_1, q_1)(1, p_2) = x$. From [Sio12, Lemma 7] it follows trivially that $t \mapsto u(tx, tq)$ is strictly increasing if $(x, q) \in \{u > \infty\}$ and $(x, q) \neq 0$; thus

$$-\infty < w(x) \leq \tilde{u}(x, p_1) = u(a_1, q_1) < u(t_1(a_1, q_1)) \leq \tilde{u}(x, p_2). \quad (19)$$

Moreover if $q \leq 0$ then $(x - qp_2, q)(1, p_1) = x + q(p_1 - p_2) \leq x$ and so $u(x - qp_2, q) \leq \tilde{u}(x, p_1)$; it follows that (19) implies that $q \neq \tilde{q}(x, p_2)$, concluding the proof of item one. Item two follows analogously. \square

PROOF OF THEOREM 1 As shown in [Sio12, Corollary 8], $\mathcal{P}(x, 0)$ is the image of the set $\partial u(x, 0)$ through the ‘perspective function’ $(y, r) \mapsto r/y$. Since $\partial u(x, 0)$ is compact (see [BNO03, Proposition 4.4.2]) and convex, this implies that $\mathcal{P}(x, 0)$ is also compact and convex¹³ set; thus, there exist $a \leq b$ such that $[a, b] = \mathcal{P}(x, 0)$. The inclusion $\mathcal{P}(x, 0) \subset \mathcal{P}$ is proved in [Sio12, Theorem 1]; since \mathcal{P} is open and convex (by Lemma 5), $\mathcal{P} = (\underline{p}, \bar{p})$ for some $\underline{p} < a \leq b < \bar{p}$. The continuity of \tilde{q} and \tilde{u} and the asymptotic results follow from Theorem 2. By definition $p \in \mathcal{P}(x, 0)$ iff $\tilde{q}(x, p) = 0$, and so Lemma 7 implies that $\tilde{q}(x, \cdot)$ is strictly positive on (\underline{p}, a) , it equals zero on $[a, b]$, and it is strictly negative on (b, \bar{p}) . Also, by definition $\tilde{u}(x, p) \geq u(x, 0)$ with equality holding iff $p \in \mathcal{P}(x, 0)$; so \tilde{u} is constant on $\mathcal{P}(x, 0)$, and Lemma 7 implies that it is strictly increasing on (\underline{p}, a) and is strictly decreasing on (b, \bar{p}) . \square

6 An example of a non-convex dependence on prices

In the case of an exponential utility U , the function $p \mapsto \tilde{u}(x, p)$ is strictly convex (see [IJS05, Theorem 3.1]). However, we will now show that, in our general framework, unfortunately convexity does not hold. To do so, we will first build a counter-example using a ‘utility function’ U which is not differentiable at $x = 1, 3$, is convex but not strictly convex, and which satisfies otherwise all the properties we assume for a utility function. To obtain an example starting with a (differentiable, strictly

¹³Because the perspective function sends convex sets to convex sets, as proved in [BV04, Section 2.3.3]. Alternatively, convexity can also easily be proved directly from the definition of $\mathcal{P}(x, q)$

convex) utility function, it is then enough to consider a sequence of (differentiable, strictly convex) utility functions U_n converging to U *uniformly*, and to note that trivially the corresponding maximal expected utilities \tilde{u}_n converge uniformly to \tilde{u} ; and so, since $\tilde{u}(x, \cdot)$ is not convex, for some n also $\tilde{u}_n(x, \cdot)$ must be not convex.

In our counter-example we will consider a function U which is affine in $[1/2, 1]$, $[1, 3]$ and $[3, 4]$, which satisfies

$$U(1) = 0, \quad U'(1-) = 9, \quad U'(1+) = 1, \quad U'(3+) = 1/9,$$

and which is extended to $(0, \infty)$ in a way that U is strictly increasing, convex, differentiable at all points other than $x = 1, 3$, and satisfies Inada conditions (2). We consider a market with no stocks, and one derivative f with distribution given by $P(f = 1) = 2/3$ and $P(f = -1) = 1/3$; the interval of its arbitrage-free prices is clearly $\mathcal{P} = (-1, 1)$. Then, the expected utility $u(x, q)$ equals $(2U(x+q) + U(x-q))/3$, and so the maximal expected utility $\tilde{u}(x, p)$ at $x = 2$ is given by

$$\tilde{u}(2, p) = \max_q \frac{2}{3}U(2 - qp + q) + \frac{1}{3}U(2 - qp - q). \quad (20)$$

The maximizer $q = \tilde{q}(2, p)$ of (20), if $p > 0$, is given clearly by $2 - qp - q = 1$, and so $\tilde{q}(2, p) = 1/(1 + p) = 1 - p + o(p)$ and since $U(x) = x - 1$ on $[1, 3]$, we obtain

$$\tilde{u}(2, p) = \frac{2}{3} \left(1 + \frac{1-p}{1+p} \right) = \frac{2}{3} \left(1 + (1 - 2p) + o(p) \right) = \frac{2}{3} \left(2 - 2p + o(p) \right).$$

Analogously, if $p < 0$, \tilde{q} is given clearly by $2 - qp + q = 3$, i.e., $\tilde{q}(2, p) = 1/(1 - p) = 1 + p + o(p)$ and so

$$\tilde{u}(2, p) = \frac{1}{3} \left(5 - \frac{1+p}{1-p} \right) = \frac{1}{3} \left(5 - (1 + 2p) + o(p) \right) = \frac{2}{3} \left(2 - p + o(p) \right).$$

So, we obtain that

$$\partial_p \tilde{u}(2, 0+) = -2 < -1 = \partial_p \tilde{u}(2, 0-),$$

which shows that $\tilde{u}(x, \cdot)$ is not convex.

7 Differentiability

We will here prove the differentiability in p of the value function; simple examples show that the largest feasible position is not a differentiable function of the derivative's prices. The following differentiability result is trivial, but useful.

Remark 8 *Under the assumptions of Theorem 2 the function*

$$\begin{aligned} (0, \infty) \times \mathcal{P} &\longrightarrow \mathbb{R} \\ (y, p) &\longmapsto \tilde{v}(y, p) \end{aligned}$$

is continuously differentiable, and its derivatives satisfy $\nabla_p \tilde{v}(y, p) = y \nabla_r v(y, yp)$ and $\partial_y \tilde{v}(y, p) = (\partial_y v + p \nabla_r v)(y, yp)$.

PROOF Since the function v defined in (12) is convex and differentiable (see [HK04, Lemma 3]) and so continuously differentiable (see [Roc70, Theorem 25.5]), the thesis follows from the identity $\tilde{v}(y, p) = v(y, yp)$. \square

To facilitate the proof of Theorem 3 we state without proof the following real-analysis lemma.

Lemma 9 *Let f, g, h be real valued functions defined on some open set $G \subseteq \mathbb{R}^n$, and assume that $f(z) \leq g(z) \leq h(z)$ for all $z \in G$, with equality at some $z = \bar{z} \in G$. Then, if g is differentiable at $z = \bar{z}$, it has gradient $\nabla g(\bar{z}) = \nabla h(\bar{z})$. Moreover, g is differentiable at $z = \bar{z}$ if either of the following conditions is satisfied:*

1. *g is convex, and h is differentiable at $z = \bar{z}$.*
2. *Both f and h are differentiable at $z = \bar{z}$.*

Moreover, we will need the following lemma, a special case of which is the well known fact that the function $w = u(\cdot, 0)$ is differentiable. We recall that in general the function u is not differentiable (see [JHS05]).

Lemma 10 *Under the assumptions of Theorem 1, the function $t \mapsto u(t(x, q))$ is differentiable on $t > 0$ if $(x, q) \in \mathcal{K}$.*

PROOF Since $(x, q) \in \mathcal{K}$, the thesis follows from [Roc70, Theorem 23.4] once we prove that, whenever $(y_i, r_i) \in \partial u(x, q)$, $i = 1, 2$, the equality $xy_1 + qr_1 = xy_2 + qr_2$ holds. The latter follows from the equality $xy_i + qr_i = \mathbb{E}[Y_T(y_i, r_i)(X_T(x, q) + qf)]$, since $Y_T(y_i, r_i) = U'(X_T(x, q) + qf)$ does not depend on i (for the last two identities, see [HK04, Theorem 1]). \square

We remark that the reason why the previous proof does not work when $(x, q) \in \partial \mathcal{K}$ is that the assumptions of [Roc70, Theorem 23.4] are not satisfied. The assumptions of this theorem however cannot be weakened: in fact, it is easy to show that the theorem fails for points not in the (relative) interior of the domain¹⁴.

Note that in the case of power utilities, some explicit computations are possible; indeed if $U(x) = x^\alpha/\alpha$ for some non-zero $\alpha \in (-\infty, 1)$, then $V(y) = -y^\beta/\beta$ for $\beta := \alpha/(\alpha - 1)$. By homogeneity it easily follows that

$$\tilde{v}(y, p) = \frac{-y^\beta}{\beta}(-\beta\tilde{v}(1, p)),$$

and since $-\beta\tilde{v}(1, p) > 0$ the bi-conjugacy relationships yield

$$\tilde{u}(x, p) = \frac{x^\alpha}{\alpha}(-\beta\tilde{v}(1, p))^{1-\alpha}. \quad (21)$$

¹⁴Indeed, consider the convex function given by $g(x, y) := \max(1 - \sqrt{x}, |y|)$ for $x \geq 0$, and $g(x, y) = \infty$ if $x < 0$; then $\partial g(1, 1) = (-\infty, 0] \times \{1\}$, yet the function $h(y) := g(0, y) = \max(1, |y|)$ is not differentiable at $y = 1$ even if $(a, b)(0, 1)$ is constant over $(a, b) \in \partial g(1, 1)$.

REFERENCES

PROOF OF THEOREM 3. As explained in [HK04, Remark 6], we can assume without loss of generality that (9) holds, so we can use Remark 8. Denote by $y^* := \tilde{y}(x^*, p^*)$, $q^* := \tilde{q}(x^*, p^*)$ the optimizers for $p = p^*$.

If U is a power utility, the thesis follows from equation (21).

To prove that $p \mapsto \tilde{u}(x^*, p)$ is differentiable if it is convex, simply apply Lemma 9 to the functions $g(p) := \tilde{u}(x^*, p)$, $h(p) := \tilde{v}(y^*, p) + x^* y^*$.

To prove differentiability under the other hypothesis, define¹⁵ $t(p) > 0$ as

$$t(p) := \frac{x^*}{x^* + q^*(p - p^*)} \quad , \text{ so that } \quad t(p)(x^* - q^* p^*, q^*)(1, p) = x^*.$$

Taking Lemma 10 into account, applying Lemma 9 to the functions

$$f(p) := u(t(p)(x^* - q^* p^*, q^*)), \quad g(p) := \tilde{u}(x^*, p), \quad h(p) := \tilde{v}(y^*, p) + x^* y^*$$

proves that $p \mapsto \tilde{u}(x, p)$ is differentiable at $p = p^*$. Moreover, whenever \tilde{u} is differentiable, Lemma 9 shows that

$$\nabla_p \tilde{u}(x^*, p^*) = \nabla_p \tilde{v}(y^*, p^*) = y^* \nabla_r v(y^*, y^* p^*),$$

which by [Sio13, Theorems 1 and 2] equals

$$\nabla_p \tilde{u}(x^*, p^*) = -q^* \partial_x \tilde{u}(x^*, p^*).$$

Since \tilde{q} and $\partial_x \tilde{u}$ are continuous function of (x, p) (by Theorem 2), the proof is finished. \square

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¹⁵Note that $t(p)$ is defined and strictly positive for p close to p^*

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